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Topological Geometrodynamics. II. Semiclassical Theory

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Topological geometrodynamics (TGD) is an attempt to a unified description of fundamental interactions based on the assumption that physically allowed space-times are representable as submanifolds of the space, which is a Cartesian Product of Minkowski space (or possibly of its light cone) and of CP_2, the complex projective space of two complex dimensions. This paper, which is the second one in the series intended for the presentation of TGD, is devoted to the dynamical considerations at semiclassical level. The concept of the classical space-time is formulated as a length-scale-dependent notion starting from the idea that particles correspond to topological inhomogeneities pointlike in the length scale considered. A semiclassical description of matter using Einstein-Yang-Mills-type effective action is studied. The requirement that effective action reproduces Maxwell electrodynamics at long length scales is shown to lead to a rather deep revision of the basic ideas concerning the description of electroweak and color interactions.

1. INTRODUCTION

This paper is a second one in a series devoted to an attempt of constructing a theory of fundamental interactions based on the following assumptions:

(1) The physically "allowed" space-times are representable as submanifolds of some space $H = V \times S$, where V denotes either Minkowski space or its light cone; S is some compact space with spacelike metric.

(2) The metric of X^4 is induced from the metric of the space H (Eisenhart, 1964).

(3) The isometries of the space H act as symmetries of the theory; the conservation laws of four-momentum and angular momentum follow from the isometries of the M^4 factor.

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In the first paper of the series we introduced the basic theoretical framework, which underlies this paper. The basic ideas of the TGD approach are (Pitkänen, 1981, 1983):

(1) The isometries of the space S correspond to color isometries; the space CP_2 turns out to be a promising candidate for the space S (Eguchi et al., 1980; Gibbons and Pope, 1978; Hawking and Pope, 1978).

(2) Both color and gravitational interactions couple to the isometry charges of the space H; as a consequence one can relate gluonic field variables to the induced metric in X^4 and derive a relationship between color and gravitational couplings.

(3) The electroweak gauge potentials are obtained by inducing spinor connection of the space S to the surface X^4; thus a geometrization of electroweak gauge potentials (Weinberg, 1967; Salam, 1968; Glashow, 1961) is achieved.

(4) The notions of particle and 3-space are generalized. Practically any action constructable from local coordinate invariants of X^4 allows stringlike objects, that is, surfaces, which are Cartesian products of a minimal surface of Minkowski space and of a geodesic sphere of the space S, as its extremals. The identification of these stringlike objects as hadrons and the ensuing generalization of the string model (Nambu, 1970; Anderson et al., 1983; Chew and Rosenzweig, 1978; Schwartz, 19) can be summarized into the statement that free particle corresponds to a compact 3-manifold.

As a consequence one obtains a topological description of particles and particle reactions. In particular, a topological explanation of the family replication phenomenon (Fritsch and Minkowski, 1975; Georgi, 1975) emerges. This assumption reduces the unification problem to the level of one fermion family in the sense that the quantum numbers to be explained by the geometric properties the space H are those associated with a single-particle generation.

Moreover, when the dimension of the space H is small enough, a mechanism generating the classical 3-space exists. Classical 3-space with matter corresponds to an approximately flat 3-surface of macroscopic size to which the particle like 3-manifolds are "glued" as topological inhomogeneities.

It was found that the space CP_2, the projective space of two complex dimensions (Eguchi et al., 1980; Gibbons and Pope, 1978; Hawking and Pope, 1978) is a unique candidate for the space S once the above ideas are accepted.

Two basically different dynamical scenarios turned out to be possible in the CP_2 framework. In the first senario, the leptonic spinors are the only elementary fermion fields and quarks correspond to leptons in pseudo triplet color partial waves. The fractional electromagnetic charges result from the anomalous hypercharge $Y_A = 2Q_{cm}$ associated with these partial waves. In the variants of the second scenario either leptons and quarks or only quarks are elementary fermions.

The following results related to the description of the electroweak interactions in TGD framework were found.

(1) TGD framework provides a geometrization of the Higgs field concept.

(2) The transition to the so-called unitary gauge, where the charged components of the Higgs field vanish, shows that the requirement "mass splitting occurs and neutrinos are massless" leads to a unique *CP*-breaking term in Dirac action.

(3) A dualism exists between the phenomena of symmetry breaking and confinement. The confinement occurs in the sense that the field quantities in the unitary gauge (uniquely defined, when Higgs field is nonvanishing) can be regarded as $SU(2)_L$ singlets build as local composites of the gauge fields, spinor fields and of the Higgs field normalized to unity. Thus physical particles belong only apparently to $SU(2)_L$ gauge multiplets and it is easy to understand the mass splittings between them.

Alternatively, one can say that symmetry breaking occurs since physical particles correspond to field quantities in a physically preferred gauge, the unitary gauge.

In this paper we study the dynamics of the theory in semiclassical level. In the first section we formulate the concept of classical space time as a length-scale-dependent concept and introduce the concept of the length-scale-dependent effective action (Iztykson and Zuber, 1980; Amit, 1975). Moreover, we derive under certain conditions Einstein equations (Migner et al., 1975; Adler et al., 1975) for the #-condensed matter.

In the second section we formulate the constraints that the length-scale-dependent bosonic effective action should satisfy (dependence on length scale through the coupling constants only, locality, presence of only first derivatives, minimally broken conformal invariance) and show that the symmetry-breaking Yang–Mills action satisfies these constraints.

We will show that solutions of Einstein–Yang–Mills equations define a set of extremals of the effective action. Explicit expression for the various terms appearing in the action will be derived and the general properties of the action will be considered. Finally, the extremals of this action will be studied in detail. The motivation for this rather technical section is that the results obtained will be used in the remaining sections of the paper.

In the third section we shall consider the general features of the coupling constant evolution. It will be found that the requirement that effective action reproduces Maxwell electrodynamics at long length scales leads to $SU(2)_L$ confinement picture: at the limit of long length scales Weinberg angle [and the inverse of $SU(2)_L$ coupling] must vanish. This scenario is in accordance with the previous observation that the concepts of $SU(2)_L$ confinement and $SU(2)_L$ symmetry breaking are dual to each other.

Also it will be found that the approximate canonical invariance of the effective action in the "Maxwellian phase," in which the $SU(2)_L$ coupling diverges, provides an explanation for color confinement (Susskind and Kogut, 1976; Brander, 1981). Thus the phenomena of color and $SU(2)_L$ confinement seem to be closely related in TGD framework. Of course, the possibility to derive color confinement by studying Maxwell action (1) can be regarded as a demonstration of unifying power of the TGD approach.

Finally, the properties of the theory at short length scales will be considered and it will be found that a transition to "minimal surface phase" probably occurs. In this phase particlelike space-times (minimal surfaces) with vanishing Higgs field dominate the functional integral and the concept of continuous macroscopic spacetime becomes meaningless.

In the fourth section we consider the cosmological consequences of TGD approach. We show that the choice $H = M^4_+ \times CP_2$, where M^4_+ is the light cone of Minkowski space provides a possible solution to the horizon problems (Weinberg, 1972) of the standard cosmologies and that the cosmologies imbeddable to H have mass density smaller than the so-called critical mass density (Fall and Lynden-Bell, 1981) based on extremals of the effective action, which we call spiral cosmic strings [analogous objects are encountered also in the context of grand unified theories (Zeldovich, 1980)].

NOTATIONS

Symbol	*Meaning*
$H = V \times S$	Imbedding space, which is Cartesian product of V and S
M^4/M^4_+	Minkowski space/light cone of Minkowski space
CP_2	Complex projective space of complex dimension 2
$h^k/m^k/s^k$	Coordinates for space $H/M^4/S$
$\xi^k, \bar{\xi}^k, k = 1, 2$	Complex coordinates for CP_2
x^α/ξ^H	Coordinates for the interior/boundary component of a submanifold X^n
$h^k_{1\alpha}/m^k_{1\alpha}/s^k_{1\alpha}$	Partial derivatives of the coordinate variables of $H/M^4/S$ with respect to the coordinate variables of X
$h_{kl}/m_{kl}/s_{kl}$	Components of the metric tensor for $H/M^4/S$
e^A_k	Components of the vielbein in H

$V_k/B_k/A_k$	Components of vielbein connection/Kahler potential in CP_2/spinor connection in H
$V_{kl}/J_{kl}/F_{kl}$	Curvature form of vielbein connection/Kahler form/curvature form of spinor connection
$g_{\alpha\beta}=h_{kl}h^k_{1\alpha}h^l_{1\beta}$	Induced metric in X^n
$V_\alpha/B_\alpha/A_\alpha$	Induced vielbein connection/Kahler potential/spinor connection in X^n
$H^k_{\alpha\beta}=D_\alpha h^k_{1\beta}$	Second fundamental form for X^n
$H^k=g^{\alpha\beta}H^k_{\alpha\beta}$	Trace of the second fundamental form
Γ_k	Gamma matrices for the space H
$\Gamma_\alpha=\Gamma_k h^k_{1\alpha}$	Gamma matrices for X^n
$\gamma_A/\Sigma_{AB}=[\gamma_A,\gamma_B]/4$	Flat space gamma/sigma matrices for space H
$\tilde{\Gamma}_k/\tilde{\Gamma}_\alpha$	Modified gamma matrices of the space H/X^n
D_k/D_α	Covariant derivative in H/X^n
X^n	n-dimensional submanifold of H
$\text{Int}\, X^n$	Interior of X^n
$\delta_i X^n$	ith boundary component of X^n

2. TGD DESCRIPTION OF MATTER AND SPACE-TIME

In this section we shall formulate the concepts of classical space-time and matter in TGD. In the first subsection we formulate a length scale dependent definition of the classical spacetime as an extremal of a length scale dependent effective action. In the second subsection we formulate the TGD description of matter, derive Einstein equations for the "classical matter" and derive variational principle having as among its extremals the spacetimes satisfying Einstein equations.

2.1. A Classical Space-Time as Length-Scale-Dependent Concept

In field theories one defines the concept of the classical field always with respect to some reference length scale L. The procedure for defining the "classical field in length scale L" is roughly the following:

(1) Cutoff procedure is defined as a procedure uniquely attaching to a given field configuration (Iztykson and Zuber, 1980; Amit, 1975):

$$\Phi(x)=\int d^dk\,\exp(ik\cdot x)\Phi(k) \tag{1}$$

the cutoff field ϕ_L, which is "trivial" in the length scales smaller than L. A possible definition of the cutoff procedure is the following one:

$$\Phi(x)\to\phi_L(x)=Z(L)\int_{k>1/L} d^dk\,\exp(ik\cdot x)\phi(k) \tag{2}$$

The procedure means simply the dropping of all Fourier components with wave vector satisfying the condition $k > 1/L$ and renormalization of the field.

(2) The concept of the length-scale-dependent effective action S_L is defined. Roughly, S_L is obtained via the following procedure:

(a) The quantity $\exp(iS)$, where S is the bare action defining the theory is averaged over all field configurations having the given cutoff configuration:

$$A_L \exp(iS_L) = \int_{k>1/L} \prod d\Phi(k) \exp(iS) \tag{3}$$

Here the quantities A_L and S_L are real and A_L is absorbed into the integration measure over the field configurations trivial in length scales smaller than L.

(b) The phase S_L is identified as the length-scale-dependent effective action.

Provided the saddle point approximation can be applied to the calculation of remaining functional integral over Fourier components satisfying the condition $k < 1/L$, the extremal of S_L can be identified as classical field in length scale L.

The generalization of the cutoff procedures to TGD context should be coordinate invariant and thus should have purely geometric description. If one can define a procedure, which associates a unique cutoff space time "trivial" in length scales smaller than L then one obtains a cutoff procedure for the induced fields as a byproduct. The existence of the cutoff procedure is however questionable. The following approach seems more natural.

(1) One can define what the addition or subtraction of a detail smaller than L the space-time means. One simply performs a deformation of space-time, which is contained in a set of M^4 having volume L^4. This deformation can lead to a different topology. Observe that this procedure is well defined also in the configuration space consisting of spacelike 3-manifolds.

(2) It is natural to define two space-times to be equivalent in length scale L if they can be obtained from each other through a finite number of deformations of size smaller than L. The equivalence class X_L^4 formed by equivalent space-times corresponds to the cutoff field configuration of the conventional field theories. The set of the equivalence classes, which corresponds to the space of cutoff fields, might be called the configuration space modulo details smaller than L.

(3) The effective action in length scale L can be defined as a functional defined in the set of these equivalence classes by averaging the bare action over the four surfaces belonging to the equivalence class X_L^4:

$$A_L \exp(iS_L) = \int_{X^4 \in X_L^4} DX^4 \exp(iS) \tag{4}$$

As a consequence the functional integral defining the theory can be expressed in the form

$$\int DX_L^4 \exp(iS_L) \tag{5}$$

Here the factor A_L is absorbed into the integration measure over details smaller than L.

(4) It would probably be difficult to do any practical calculations in the space of the equivalence classes X_L^4. One can however extend the effective action to a functional in the original configuration space having the property that it is constant inside each equivalence class. Of course, the integration measure DX_L^4 in the functional integral must be divided with the volume of the equivalence class X_L^4.

The extremals of this action have a characteristic degeneracy resulting from its constancy inside the equivalence classes X_L^4. Any extremal of this action can be called a classical space-time trivial in the length scales smaller than L and the associated induced field quantities are the analogues of the cutoff fields of the conventional field theories.

2.2. TGD Description of Matter

The phenomenon of ♯ condensation is a mechanism, which in a certain sense generates classical space-time with matter. The essential features of the phenomenon are the following:

(1) In the presence of a surface X^4, representable as a graph of some map $M^4 \to S$ and having large enough size, the particle like 3-surfaces "collide" with these surfaces with a high probability provided the dimension of the space H is sufficiently small ($\dim H < 9$).

(2) Under certain circumstances we expect that the particles get "stuck" to the surface X^3 so that something resembling classical 3-space with particles appearing as topological inhomogenieties emerges.

(3) It seems natural to identify the many-particle states formed as bound many-particle states. This would mean the identification of gravitationally bound many particle states as well as quantum mechanical bound states (molecules, atoms, nuclei, hadrons) as ♯ condensates.

(4) If this interpretation is accepted one obtains a thermodynamical criterion for the occurrence of the ♯ condensation: particles in ♯ condensate are stable against "♯ evaporation" provided the temperature is smaller than the binding energy of the particle.

Accepting this general picture one must take the length-scale-dependent description of matter into account in three different contributions to isometry currents. These contributions correspond to the isometry currents associated with the long-range fields, with the ♯-condensed matter and with the free, non-♯-condensed particles ("vapor phase").

The form of the vacuum contribution, which physically corresponds to the contribution of fields having range larger than the length scale, is determined by the form of the length-scale-dependent effective action. We denote the corresponding tensor by $T_V^{\alpha k}$. In general the #-condensed matter creates long range fields and interacts with them so that this tensor is not divergenceless as it is for the extremals of the length-scale-dependent effective action.

The #-condensed particles of size smaller than L can be described using energy momentum tensor $T_\#^{\alpha k}$. We shall denote by the symbol $T^{\alpha k}$ the sum of these two energy momentum tensors:

$$T^{\alpha k} = T_V^{\alpha k} + T_\#^{\alpha k} \tag{6}$$

Since the total energy momentum flow must be parallel to the surface X_2^4, this tensor is parallel to the surface X_L^4 as a vector of H and thus can be written in the form

$$T^{\alpha k} = T^{\alpha\beta} h_{1\beta}^k \tag{7}$$

The associated isometry current is obtained by contracting with the infinitesimal generator of isometry.

The free particles must be described by an energy momentum tensor T^{kl} defined in H, which clearly has no counterpart in GRT. Contractions with the generators of isometries give the isometry currents.

The currents $T^{\alpha k}$ and T^{kl} are not conserved. Their divergences must however be orthogonal to the surface X_L^4 since the flow of the various quantum numbers between the condensed phase and "vapor phase" is orthogonal to the surface X_L^4.

From the conservation of isometry currents one obtains the condition

$$D_l T^{kl} = \int d^4x\, \delta(h^k, h^k(x)) D_\alpha T^{\alpha k} \tag{8}$$

Since both sides are orthogonal to the surface X_L^4 as vectors of H, the quantity

$$D_\alpha T^{\alpha k} = D_\alpha T^{\alpha\beta} h_{1\beta}^k + T^{\alpha\beta} H_{\alpha\beta}^k \tag{9}$$

must be orthogonal to X_L^4.

The term proportional to the second fundamental form satisfies this condition but the first term does not unless the tensor $T^{\alpha\beta}$ is divergenceless:

$$D_\beta T^{\alpha\beta} = 0 \tag{10}$$

By the well-known arguments (Misner et al., 1975; Adler et al., 1975) one finds that the tensor $T^{\alpha\beta}$ must be expressible as a linear combination of the metric tensor and Einstein tensor associated with the space X_L^4:

$$T^{\alpha\beta} = \kappa G^{\alpha\beta} + \lambda g^{\alpha\beta} \tag{11a}$$

Thus we have derived Einstein equations for the ordinary matter assuming only that the pointlike particles in length scale L consist of two phases: the condensed phase and vapor phase. Of course, the conservation of isometry charges is an essential ingredient in the proof.

The remaining equations should govern the interaction between the vacuum and #-condensed matter. It seems natural to describe the #-condensed matter as external Yang–Mills currents coupled to long-range fields. Thus these currents act as sources for the long-range fields so that Yang–Mills equations

$$D_\beta F^{\alpha\beta} = j_\#^\alpha \tag{11b}$$

should be satisfied.

As is well known, the ordinary Einstein equations can be derived from a variational principle. Also now the above-derived conclusions follow from a variational principle. The variational principle is obtained in an obvious manner: add to the effective action S curvature scalar part and source term describing the effects of #-condensed matter to the surrounding vacuum.

The action principle can be written in the form

$$\begin{aligned} S &= \int L\, d^4x \\ L &= (L_L + \kappa R)(-g)^{1/2} + T^\#_{\alpha\beta} g^{\alpha\beta} + \mathrm{Tr}(j_\#^\alpha A_\alpha) \\ T &= T^\#_{\alpha k} h^k_{1\beta} \end{aligned} \tag{12}$$

Here L_L is the Lagrangian associated with the long-range fields, R denotes the curvature scalar, $T^\#_{\alpha k}$ and $j_\#^\alpha$ are the energy momentum and Yang–Mills currents associated with the #-condensed matter, and A_α denotes the induced Yang–Mills connection in X_L^4.

We shall show later that when the bosonic effective action is of Yang–Mills type, the resulting field equations are solved identically by the ansatz

$$T_\#^{\alpha\beta} + T_{\mathrm{YM}}^{\alpha\beta}(-g)^{1/2} = \kappa G^{\alpha\beta}(-g)^{1/2} \tag{13}$$

$$j_\#^\alpha = D_\beta F^{\alpha\beta}(-g)^{1/2} \tag{14}$$

$$D_\alpha T_\#^{\alpha k} = -\mathrm{Tr}(D_\alpha F^{\alpha\beta} F^\gamma_\beta h^k_{1\gamma})(-g)^{1/2} \tag{15}$$

We shall later find that field equations allow also more general solutions implying a departure from the Einstein equations.

Summarizing, we have derived the Einstein equations for the ordinary matter essentially from the requirement that energy momentum is conserved and found them to be derivable from an action principle. We do not however believe that this action has any fundamental role in the definition of the theory, since it describes the ♯-condensed matter as external currents. Rather, the bosonic effective action S_L (not containing the curvature scalar) might have something to do with quantum theory.

An important departure from the GRT picture is the assumption about the presence of the matter consisting of free particles. The assumption of this form of matter is natural in TGD context and also necessary in order to obtain Einstein equations together with overall energy momentum conservation. We shall show later that this departure might solve some basic problems of the standard big bang cosmology.

3. MODELING THE BOSONIC EFFECTIVE ACTION

In the following we shall study a model for the bosonic effective action based on some simplifying assumptions such as locality and presence of only first-order derivatives. We shall derive explicit representation for the field equations describing the dynamics of the ♯-condensed matter interacting with the long-range fields and show that the solutions of Einstein equations are solutions of the equations thus obtained.

In the remaining sections we shall study the properties of the bosonic effective action in more detail; as a motivation for these rather technical considerations is that the results can be used in the subsequent sections. We derive explicit form for the action and show that it allows two non-equivalent Abelian "subtheories." An expression for the Weinberg angle in terms of the parameters appearing in the action will be derived.

In Section 3.3 we study the properties of the pure $U(1)$ action to which the bosonic effective action reduces in the limit of vanishing Weinberg angle and show that it has to a good approximation the properties of the Maxwell action and that its characteristic property is its enormous vacuum degeneracy. In addition the extremals of the bosonic effective action will be studied.

3.1. Constraints on the Choice of the Effective Action

Our basic assumptions about the properties of the bosonic effective action are the following ones:

(1) S is constructable from local invariants, the dominant terms involving only first derivatives of the coordinate variables of the space H and S is positive definite.

(2) S depends on the length scale only via the length-scale-dependent coupling parameters (gauge couplings, etc.).

(3) Effective action breaks conformal invariance minimally in the sense that the effective action contains only dimensionless coupling parameters and the covariant field quantities appearing in the action are dimensionless.

As far as we know, the most general action satisfying these conditions is symmetry breaking Yang-Mills action, which is a superposition of the two invariants. The first invariant is given by

$$I_1 = \mathrm{Tr}(F^{\alpha\beta} F_{\alpha\beta}) \tag{16}$$

Here F denotes the projection of the curvature form of the spinor connection of CP_2 and I_1 can be identified as the symmetry conserving part of the effective action associated with electroweak interactions. The second invariant is given by

$$I_2 = J^{\alpha\beta} J_{\alpha\beta} \tag{17}$$

J is the projection of the Kahler form of CP_2 identifiable as $U(1)$ gauge field and the presence of this invariant reflects the "occurrence" of electroweak symmetry breaking implying different evolution for $U(1)$ and $SU(2)_L$ couplings at low energies.

Summarizing, the effective action to be studied in the sequel is symmetry breaking YM action

$$L = -(1/4g^2) I_1 - f I_2 \tag{18}$$

where g and f are length-scale-dependent dimensionless parameters.

It is important to notice that the presence of Higgs term constructed from the components of second fundamental form and from their covariant derivatives might in principle be present in the action. We will later find that the presence of Higgs term is not excluded at sufficiently short length scales.

3.2. General Form of Field Equations

The action which we believe to give an approximate description of matter in TGD is given by

$$\begin{aligned} S &= \int L\, d^4x \\ L &= (L_L + \kappa R)(-g)^{1/2} + T^{\#}_{\alpha\beta} g^{\alpha\beta} + \mathrm{Tr}(j^{\alpha}_{\#} A_\alpha) + j^{\alpha}_{1\#} B_\alpha \qquad (19) \\ T^{\#}_{\alpha\beta} &= T^{\#}_{\alpha k} h^k_{1\beta} \end{aligned}$$

Here L_L is the Lagrangian associated with the long-range fields derived in the preceding section and R denotes the curvature scalar. $T^{\#}_{\alpha k}$, $j^{\alpha}_{\#}$, and $j^{\alpha}_{1\#}$ are the energy momentum and Yang-Mills currents associated with the #-condensed matter; they are not regarded as dynamical variables but external currents. A_α and B_α denote the induced Yang-Mills connection and Kahler potential respectively.

Field equations are most easily derived by using Lagrange multipliers for the induced field quantities $g_{\alpha\beta}$, A_α, and B_α. This is accomplished by regarding these field variables as primary dynamical variables and by adding to the action the following Lagrange multiplier terms taking care of the constraint conditions

$$S_c = \int L_c\, d^4x \tag{20a}$$

$$L_c = \mathrm{Tr}(J^\alpha(A_\alpha - A_k h^k_{1\alpha}) + J^\alpha_1(B_\alpha - B_k h^k_{1\alpha}) + T^{\alpha\beta}_1(g_{\alpha\beta} - h_{kl} h^k_{1\alpha} h^l_{1\beta}) \tag{20b}$$

Here the variables $T^{\alpha\beta}_1$, J^α, and J^α_1 are regarded as auxiliary dynamical variables.

Varying with respect to all dynamical variables one obtains the field equations, which read

$$\begin{aligned}[T^{\alpha\beta}_{\#} + T^{\alpha\beta}_V - \kappa G^{\alpha\beta}(-g)^{1/2}]H^k_{\alpha\beta} + \mathrm{Tr}((j^\alpha - j^\alpha_{1\#})F^k_l)h^l_{1\alpha} + (j^\alpha - j^\alpha_{1\#})J^k_l h^l_{1\alpha} \\ + (\mathrm{Tr}(j^\alpha F^\beta_\alpha) + j^\alpha_1 J^\beta_\alpha)h^k_{1\beta} - D_\alpha T^{\alpha k}_{\#} = 0\end{aligned} \tag{21a}$$

Here the tensor $T^{\alpha\beta}_V$ is the canonical energy momentum tensor associated with the symmetry broken Yang-Mills action

$$\begin{aligned}T^{\alpha\beta}_V = -(1/4g^2)\,\mathrm{Tr}(F^\alpha_\gamma F^{\gamma\beta} - (g^{\alpha\beta}/4)F^{\mu\nu}F_{\mu\nu})(-g)^{1/2} \\ -4f(J^\alpha_\gamma J^{\gamma\beta} - (g^{\alpha\beta}/4)J^{\mu\nu}J_{\mu\nu})(-g)^{1/2}\end{aligned} \tag{21b}$$

The currents j and j are the canonical currents associated with the Yang-Mills action and Kahler action

$$j^\alpha = (1/g^2)D_\beta F^{\alpha\beta}(-g)^{1/2} \tag{22a}$$

$$j^\alpha_1 = 4f D_\beta J^{\alpha\beta}(-g)^{1/2} \tag{22b}$$

From the form of these equations it is clear that the solutions of Einstein-Maxwell equations

$$T^{\alpha\beta}_V + T^{\alpha\beta}_{\#} = \kappa G^{\alpha\beta}(-g)^{1/2} \tag{23a}$$

$$j^\alpha = j^\alpha_{\#} \tag{23b}$$

$$j^\alpha_1 = j^\alpha_{1\#} \tag{23c}$$

$$D_\alpha T^{\alpha k}_{\#} = -(\mathrm{Tr}(j^\alpha_V F^\beta_\alpha) + j^\alpha_1 J^\beta_\alpha)h^k_{1\beta} \tag{23d}$$

form a rather general set of solutions to these equations.

The physical content of the first three equations is obvious. The last equation states that the covariant divergence of the energy momentum tensors associated with ♯-condensed matter and long-range fields have opposite covariant divergences; this is a necessary condition for Einstein equations to hold true.

Of course, the imbeddability requirement (gauge fields are induced) sets constraints on the form of the currents $j_\sharp^\alpha$ and $T_\sharp^{\alpha k}$. On the other hand, any surface defines a solution of the field equations provided the associated Einstein tensor satisfies certain reasonability conditions (energy density must be positive definite).

It is clear, that these equations allow also more general solutions: in particular solutions for which Einstein equations do not hold. We shall later consider solutions of this type.

3.3. General Features of the Action

In this Section we shall study the general properties of the effective action. After necessary preliminaries (giving explicit representations for the various tensor quantities appearing in the action) we derive expression for Weinberg angle, show the existence of two Abelian "subtheories" and show that the pure Kahler action (Weinberg angle vanishes) has the properties of Maxwell action.

3.3.1. The Explicit Representation of the Action

The induced Yang–Mills field is the projection of the curvature form F of the spinor connection (Pitkänen, 1981, 1983). In the sequel we shall assume the following general form for the spinor connection

$$A = V + B(n_+ 1_+ + n_- 1_-)/2 \tag{24}$$

Here V and B denote the vierbein and Kahler connections of CP_2, respectively. The matrices 1_+ and 1_- project to the subspace of spinor with H chirality $+1$ and -1, respectively.

Using the explicit representations of V and J (Appendix) and the definition of the invariant I_1 given by the formula (16) one obtains for the invariant I_1 the representation

$$I_1 = \Sigma_e I_1^e \, d(e) \tag{25}$$

where the factor $d(e)$ is equal to 1 or 0 depending on whether the fermions with the chirality e are elementary fermions or not.

The contribution of a single chirality is given by the expression

$$\begin{aligned} I_1^e = 8(&X((e_0 \wedge e_3)^2 + (e_1 \wedge e_2)^2) + 64(e_0 \wedge e_3, e_1 \wedge e_2) \\ &+ 2(e_0 \wedge e_1 - e_2 \wedge e_3)^2 + 2(e_0 \wedge e_2 - e_3 \wedge e_1)^2) \end{aligned} \tag{26}$$

where the coefficient of X of the term depending on the coupling to the Kahler potential is given by

$$X = 31 + n_e^2 \tag{27}$$

Here we have used the same notation for the projections of the vielbein components as for the vielbeins themselves (the product $A \wedge B$, when written explicitly gives the tensor $(A_\alpha B_\beta - A_\beta B_\alpha)/2$).

The explicit representation of the invariant I_2 is given by

$$I_2 = 4(e_0 \wedge e_3 + e_1 \wedge e_2)^2 \tag{28}$$

The curvature tensor of X^q is given by

$$R_{\alpha\beta\gamma\delta} = R_{ijkl} h^i_{1\alpha} h^i_{1\beta} h^k_{1\gamma} h^l_{1\delta} + h_{kl}(H^k_{\alpha\gamma} H^l_{\beta\delta} - H^k_{\beta\gamma} H^l_{\alpha\delta}) \tag{29}$$

has two parts. The first part is simply the projection of the curvature tensor of H and results from the curvature of the imbedding space. The second term involving second fundamental form is always present.

3.3.2. The Abelian Subtheories

In a pure gauge theory with gauge group G the restrictions of the gauge potential to a subgroup H of G defines a "subtheory" with gauge group H in the sense that field equations satisfied by the restricted gauge potentials and by those of H gauge theory are identical.

In TGD an analogous phenomenon occurs for any action constructable from local invariants. The role of the subgroup H is taken by the so-called geodesic submanifold $H_G \subset H$ with the defining property that the geodesics of H (with respect to the induced metric) are geodesics of H also. This requirement implies that the second fundamental form associated with this imbedding vanishes (Helgason, 1978):

$$H^k_{\alpha\beta} = 0 \tag{30}$$

The reason for the special dynamical role of the geodesic submanifolds is that induction procedure yields the same bosonic field quantities independently whether it is performed directly $(I: H \to X)$ or in two steps) $I: H \to H \to X)$.

It is evident that the geodesic submanifolds of $H = M^4 \times S$ of dimension larger than 1 are Cartesian products of the geodesic submanifolds of M and S, respectively, and that the geodesic submanifolds of M are hyperplanes of dimension $0, 1, \ldots, 4$.

In order to find the geodesic submanifolds of the space $S = CP_2$ we can use the fact that CP_2 is symmetric space, i.e., representable as a coset space of some group: $CP_2 = SU(3)/SU(2) \times U(1)$. For symmetric spaces (or equivalently constant curvature spaces) the geodesic submanifolds are describable in Lie-algebraic terms (Helgason, 1978).

Theorem. Let M be a symmetric space: $M = G/H$ of the group G and $H \subset G$. Identify the tangent space T of M at a given point m as a subspace of the Lie algebra of G in the usual manner (the orthogonal complement of the subspace spanned by the Lie-algebra generators of H). Let $S \subset T$ be a Lie triple system satisfying the defining condition

$$[X, [Y, Z]] \in S, \qquad \forall X, Y, Z \in S \tag{31}$$

Then S defines a geodesic submanifold of M via the exponential mapping $\exp: s \to \exp(s) \in M$.

The obvious Lie triple systems contained in CP_2 tangent space are the whole tangent space and the subspace defined by one Lie-algebra generator: the corresponding geodesic submanifolds are CP_2 itself and a geodesic line of CP_2.

The remaining Lie triple systems are easily found in the "particle representation" of $SU(3)$ Lie algebra. In the representation of $SU(3)$ Lie-algebra as meson octet (π, K, η) the tangent space of CP_2 corresponds to the space spanned by strange mesons: $T = (K^+, K^-, K^0, \bar{K}^0)$. It is easy to verify that the following two subspaces

$$(K^+, K^-) \tag{32a}$$

$$(K^0, \bar{K}^0) \tag{32b}$$

are nonequivalent [not related by an automorphism of $SU(3)$ Lie algebra)] Lie triple systems and that there are no three-element Lie triple systems.

Thus we can conclude that CP_2 allows two nonequivalent geodesic submanifolds; the corresponding Lie-triple systems correspond to two non-equivalent three-dimensional subalgebras of $SU(3)$ Lie algebra, which integrate to the groups $SU(2)$ (2×2 unitary matrices) and $SO(3)$ (3×3 orthogonal matrices) of $SU(3)$ (3×3 unitary matrices), respectively.

Convenient representatives for the geodesic submanifolds (spheres in fact) are defined by the equations

$$\text{I:}\ \xi^1 = \xi^2\ (\theta = \pi/2, \Phi = 0) \tag{33a}$$

$$\text{II:}\ \xi^1 = \bar{\xi}^2\ (\theta = \pi/2, \psi = 0) \tag{33b}$$

The nonequivalence of these submanifolds is clear from the fact that the isometries of CP_2 act as holomorphic transformations of CP_2. The vanishing of the second fundamental form is easy to verify by a direct calculation. We shall refer to these geodesic spheres as S^2_{I} and S^2_{II}, respectively.

It is advantageous to use the ordinary spherical coordinates for S_{I}^2 (S_{II}^2) instead of the coordinates (r, ψ) $[(r, \Phi)]$. The relationship between these two coordinate sets is

$$\begin{aligned} &\mathrm{I:}\ \cos\theta = 2(1+r^2)^{-1} - 1 \qquad &&\mathrm{II:}\ \cos\theta = \pm(1+r^2)^{1/2} \\ &\phi = \psi/2 &&\phi = \Phi \end{aligned} \tag{34}$$

In these coordinates the line element is given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)/4 \tag{35}$$

Both geodesic spheres have the same radius since CP_2 allows geodesic lines of one type only.

The nonvanishing components of the curvature form of the spinor connection are given by

$$R_{03} = 2R_{12} = 4e^0 \wedge e^3 = -du \wedge d\phi \tag{36}$$

and by

$$R_{02} = -R_{31} = e^0 \wedge e^2 = du \wedge d\phi/2 \tag{36b}$$

in the cases I and II, respectively.

The S_i^2 restrictions of the invariants I_1 and I_2 are proportional to the invariant

$$I = (du \wedge d\phi)^2 \tag{37}$$

A convenient representation for the proportionality constants is in the form of the matrix

$$I_k(S_i^2) = n_k^i I \tag{38}$$

The general form of this matrix is given by

$$\begin{pmatrix} n_1^{\mathrm{I}} & n_2^{\mathrm{I}} \\ n_1^{\mathrm{II}} & n_2^{\mathrm{II}} \end{pmatrix} = \begin{pmatrix} X & 1/4 \\ Y & 0 \end{pmatrix} \tag{39}$$

Here the quantities X and Y depend on the coupling to the Kahler potential

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \Sigma_e\, d(e) \begin{pmatrix} (31 + n_e^2)/2 \\ 4 \end{pmatrix} \tag{40}$$

The meaning of the factor $d(e)$ should be already familiar.

3.3.3. The Evaluation of Weinberg Angle

In the first paper of the series (and already in Pitkänen, 1983) we demonstrated that the requirement "Photon couples vectorially" leads to the coupling structure of the GWS model of the electroweak interactions

(Weinberg, 1967; Salam, 1968; Glashow, 1961); the value of the Weinberg angle remained however an undetermined parameter.

Here we want to show that the value of the Weinberg angle can be fixed uniquely by requiring that the Yang-Mills part of the effective action contains no nondiagonal terms that are terms of the form γZ^0.

To evaluate the value of the Weinberg angle we express the neutral part of the induced gauge field F_{nc}

$$F_{nc}=2R_{03}\Sigma^{03}+2R_{12}\Sigma^{12}+J/2(n_+1_+ + n_-1_-) \tag{41}$$

where we have

$$\begin{aligned} R_{03}&=2(2e^0\wedge e^3+e^1\wedge e^2)\\ R_{12}&=2(e^0\wedge e^3+2e^2\wedge e^2)\\ J&=2(e^0\wedge e^3+e^1\wedge e^2)\end{aligned} \tag{42}$$

in terms of the fields γ and Z^0

$$F_{nc}=\gamma Q_{\text{em}}+Z^0(I_L^3-pQ_{\text{em}}) \tag{43a}$$

$$p=\sin^2\theta_w \tag{43b}$$

Here we have

$$\begin{aligned} Q_{\text{em}}&=\Sigma^{12}+(n_+1_+ + n_-1_-)/6\\ I_L^3&=(\Sigma^{12}-\Sigma^{03})/2\end{aligned} \tag{44}$$

Evaluating the expressions (43) and (44) one obtains for γ and Z^0 the expressions

$$\begin{aligned} \gamma&=3J+pR_{03}\\ Z^0&=2R_{03}\end{aligned} \tag{45}$$

For the Kahler field one obtains

$$J=(\gamma-pZ^0)/3 \tag{46}$$

Expressing the neutral part of the symmetry broken Yang-Mills action in terms of γ and Z^0 one obtains for the coefficient of γZ^0 cross term the expression

$$X=-K/2g^2+fp/18 \tag{47a}$$

$$K=\text{Tr}(Q_{\text{em}}(I_L^3-pQ_{\text{em}})) \tag{47b}$$

In the general case the value of the coefficient K is given by

$$K=\Sigma_e(1-(18+2n_e^2)p/9)\,d(e) \tag{48}$$

where the sum runs over the H chiralities $(d(e))$ equals to 1 or 0 depending on whether elementary fermions with the corresponding chirality are assumed to be present in the theory.

The cross term vanishes provided the value of the Weinberg angle is given by

$$p = \sin^2 \theta_w = g\Sigma_e d(e)/(fg^2 + \Sigma_e d(e)(18 + 2n_e^2)) \qquad (49)$$

In the scenario in which only leptons are elementary fermion $(n_- = 3)$ the value of the Weinberg angle is given by

$$p = 1/(4 + fg^2/9) \qquad (50)$$

The bare value of the Weinberg angle in this scenario is equal to 1/4 and is remarkably close to the measured value of this parameter $(p = 0.23\ldots)$. Since this scenario is the simplest the result is rather encouraging. In other scenarios $((n_+, n_-) = (3, 1)$ or $(0, 1))$ the values of the Weinberg angle are larger but differ somewhat from the value 3/8 encountered in many GUT's (Fritsch and Minkowski, 1975; Georgi, 1975).

3.3.4. Properties of the Kahler Action

The properties of the Kahler action are interesting since the requirement that effective action produces Maxwell electrodynamics at long length scales might be satisfied provided the effective action reduces to a pure Kahler action at this limit; this of course implies the vanishing of the Weinberg angle at this limit and thus a phenomenon, which might be called $SU(2)_L$ confinement.

In what follows we shall show that (1) any Maxwell field is representable locally as an induced Kahler field, (2) Kahler action is characterized by an enormous vacuum degeneracy, and (3) the canonical transformations of CP_2 leaving the Kahler form of CP_2 invariant are approximate symmetries of the $U(1)$ action and the action of a canonical transformation on covariant Kahler field is equivalent to the action of a $U(1)$ gauge transformation.

The local representability of an arbitrary Maxwell field defined in M as an induced Kahler form follows from the representability of any $U(1)$ gauge potential $A(m)$ in the form

$$A = \Sigma_k P_k \, dQ^k, \qquad k = 1, 2 \qquad (51a)$$

Here P_k and Q^k, $k = 1, 2$ are some functions of the Minkowski coordinates (Gliozzi, 1978). This property is of course shared by the Kahler potential B:

$$B = \Sigma_k p_k \, dq^k, \qquad k = 1, 2 \qquad (51b)$$

as can be checked directly from the representation of the Kahler potential in the standard gauge.

Clearly, the map

$$(p_k, q^k) = (P_k, Q^k) \tag{52}$$

defines the required local imbedding of a given Maxwell field in M^4 as a four surface, representable as a graph of a map $M^4 \to CP_2$.

The imbedding is not unique; any canonical transformation of CP_2 leaving the Kahler form J invariant leaves also the induced $U(1)$ gauge field invariant and thus its action is equivalent to that of a $U(1)$ gauge transformation.

The vacuum degeneracy is not a property of a pure $U(1)$ action only; already the Yang–Mills action has quite a large vacuum degeneracy. Regardless of the values of the coupling parameters all the surfaces $X^4 \subset M^4 \times D^1$, where D^1 is an arbitrary one-dimensional submanifold of CP_2, are vacuum extremals of pure Yang–Mills action.

The reason for this circumstance is that the induced Yang–Mills field vanishes identically (because of the antisymmetry the projection of Yang–Mills field to any one-dimensional submanifold vanishes).

The vacuum degeneracy associated with the pure Kahler action is however much larger; any surface $X^4 \subset M^4 \times Y^2$, where Y^2 is a two-dimensional submanifold of CP_2 with the property that the projection of the Kahler form on Y^2 vanishes, is a vacuum extremal. Clearly, the Kahler form is pure gauge on these submanifolds.

One can determine these surfaces using the canonical representation of the Kahler potential. For example the surfaces representable as maps

$$P_k = \partial F / \partial Q^k \tag{53}$$

where F is an arbitrary function of the variables Q^k, defines a surface of required type. From the vacuons of type I and II, "elementary vacuons," one can build more complicated vacuum extremals by gluing them together along their boundaries. Of course, the obvious continuity conditions must hold true on the boundaries.

Of course these solutions are not vacuum extremals of the whole action. Rather, canonical transformations act as dynamical symmetries of the Einstein–Maxwell action provided the energy momentum tensor of the #-condensed matter is allowed to change in the transformation so that Einstein equations remain true. Thus these symmetries transform electromagnetically equivalent dynamical evolutions to each other.

3.4. About the Extremals of the Effective Action

In the sequel we shall show that the effective action allows as its extremals (1) Schwarzschild- and Reissner–Nördström exterior metrics

(Misner et al., 1975; Adler et al., 1975), (2) several types of stringlike objects, (3) a rather large family of massless fields, (4) minimal surfaces with vanishing Higgs field.

3.4.1. Imbedding Certain Metrics

Both Schwarzschild- and Reissner-Nördström exterior metrics are imbeddable as vacuum extremals of the pure Kahler action to $M^4 \times S^2_{\mathrm{II}}$. Of course, the Reissner-Nördström solution defines a nonvacuum solution of the whole action in the sense that the energy momentum tensor of ♯-condensed matter is nonvanishing. The Reissner-Nördström metric can be inbedded also to $M^4 \times S^2_{\mathrm{I}}$; the requirement that the energy momentum density associated with the Kahler action is smaller than the total energy density gives an upper bound for gravitational constant.

We shall use the ordinary spherical coordinates (θ, ϕ) for S^2_i; standard spherical coordinates $(m^0, r_M, \theta_M, \phi_M)$ for M^4 and coordinates (x^0, r, θ, Φ) for X^4.

The imbedding can be found via the following ansatz:

$$\begin{aligned} &u = \cos\theta = g(r), \qquad && m = \lambda x^0 + h(r) \\ &\phi \omega x^0 + f(r), && (r_M, \theta_M, \phi_M) = (r, \theta, \Phi) \end{aligned} \tag{54}$$

The interesting components of the induced metric are

$$\begin{aligned} g_{00} &= -U\omega^2 + \lambda^2 \\ g_{0r} &= h' - U\omega f' \\ g_{rr} &= -1 - U(f')^2 + (g')^2 \\ U &= R^2 \sin^2 \theta \end{aligned} \tag{55}$$

The stationarity condition $g_{0r} = 0$ fixes the function $h(r)$ apart from a constant

$$h' = Uf'\omega / 4R\lambda \tag{56}$$

The parameter λ is fixed by the boundary condition $g_{00}(\infty) = 1$:

$$\lambda^2 = 1 + U(\infty)\omega^2 \tag{57}$$

The interesting components of the Reissner-Nördström metric are given by

$$g_{00} = -1/g_{rr} = 1 - a/r - b/r^2 \tag{58a}$$

$$(a, b) = (2GM, G\pi q^2) \tag{58b}$$

The charge parameter q is defined so that the energy density associated with the $U(1)$ field is given by

$$T_0^0 = q^2/2r^4 \tag{58c}$$

Combining the equations (55) and (58) one finds

$$U = [a/r - b/r^2 + U(\infty)\omega^2]/\omega^2 \tag{59}$$

The equations (55), (57), and (59) fix the function f apart from an integration constant.

Some remarks concerning the properties of the imbedding are in order.

(1) The imbedding is not defined for the values of the radial coordinates $r > r_C$; r_C is defined by the condition

$$U(r_C) = 0 \tag{60}$$

The special case $U(\infty) = 0$ gives an upper bound for r:

$$r_C < \pi q^2/2M \tag{61}$$

which is of the order of the Compton length of the particle if the charge parameter is of the order of unity.

The gauge charge associated with the solution is nonvanishing; for the standard representative only the W component of the gauge field is nonvanishing. In fact a mere $1 - O(1/r)$ behavior of the time component of the metric tensor together with the stationarity requirement implies a nonvanishing gauge charge. In fact, for the Schwarzschild solution satisfying the condition $U(\infty) = 0$ the gauge field has $(1/r)^{1/2}$ behavior so that the gauge charge is infinite!

The result just obtained is not a contradiction provided the action becomes a pure Kahler action at long length scales); since only the photon appears in the effective bosonic action the gauge charges associated with $W^{\pm}$ and Z^0 have no physical significance. Moreover, the infinite value of the gauge charge for Schwarzschild solution could be interpreted in terms of symmetry breaking.

The imbedding of Reissner–Nördström metric to $M^4 \times S_1^2$ is also possible. If one poses the natural requirement that the energy density associated with the #-condensed matter is positive definite, one obtains an upper bound for the magnitude of the gravitational constant from the equation

$$\rho_V \leq q^2/2r^4 \tag{62a}$$

using the explicit expression for

$$\rho_V = fJ^{\alpha\beta}J_{\alpha\beta} = fG\pi q^2/8R^2r^4 \tag{62b}$$

and the equation

$$f = g/16\pi\alpha \tag{63}$$

resulting from the relationship between Kahler field and Maxwell field ($\gamma = 3J$).

The resulting inequality is

$$G \leq 64\alpha R^2/9 \tag{64}$$

and combined with the relationship between color and gravitational constants ($\alpha_S = 4G/R^2$) gives upper bound

$$\alpha_S \lesssim 256\alpha/9 \tag{65}$$

for color coupling strength.

3.4.2. Stringlike Extremals

Almost any effective action constructable from local invariants allows stringlike extremals

$$X^4 = X^2 \times Y^2 \subset M^4 \times CP_2$$

where Y^2 is a submanifold of CP_2, typically geodesic sphere, and X^2 is a surface of M^4, usually minimal surface.

The stringlike objects can be divided into two classes, which are (1) Einsteinian strings satisfying Einstein equations and (2) minimal strings; the simplest representatives of this class being surfaces, for which the surfaces X^2 and Y^2 are minimal surfaces.

Consider first the Einsteinian strings of type $X^2 \times Y^2$. From the definition of Einstein tensor in terms of Ricci tensor $R^{\alpha\beta}$ and from vanishing of the Einstein tensor for two-dimensional manifolds it follows that Einstein tensor is representable in the form

$$G^{\alpha\beta} = -[g^{\alpha\beta}(X^2)R(Y^2) + g^{\alpha\beta}(Y^2)R(X^2))/2 \tag{66}$$

If Einstein equations hold one obtains for the energy momentum of the Einsteinian strings

$$P^k = k\int g^{0\beta} m^k_{1\beta}\sqrt{g}\, dx \tag{67}$$

Here the string tension k is defined as

$$k = (1/16\pi G)\int R\sqrt{g}\, d^2x \tag{68a}$$

As is well known, for two-dimensional manifolds the integral of the curvature scalar is a topological invariant giving the so-called Euler characteristic of the two-manifold (Douglas, 1939)

$$\int R\sqrt{g}\, d^2x = 4\pi(1-g) \tag{68b}$$

Here the integer g is the genus of the 2-manifold (number of handles attached to two sphere. If the surface Y^2 has boundary one must add a boundary term to the action in order to obtain analogous result (Pitkänen, 1983); the generalization of the result to the more general case is obtained via the replacement

$$g \to g + n/2 \tag{68c}$$

where n is the number of holes in Y^2.

The vanishing of Einstein tensor in two dimensions reflects this fact. From the quantization rule it follows that the curvature scalar is locally a total divergence and thus the associated variational principle giving rise to Einstein equations must be identically satisfied.

An important result is that the integral of the curvature scalar is nonnegative only for a rather limited set of 2-topologies if Einstein equations hold true (sphere, sphere with one or two holes, and torus if orientability is assumed). Thus it is clear that Einstein equations do not make sense for all topologies unless the assumption about the positivity of the total energy is not given up.

Einstein equations fail to be satisfied also if the contribution of the symmetry broken Yang-Mills action is so large that the contribution of $\sharp$-condensed matter to energy momentum tensor cannot be positive for Einstein equations to be satisfied.

Because of their enormous string tension Einsteinian strings resemble the so-called cosmic strings of the grand unified theories (Fritsch and Minkowski, 1975; Georgi, 1975). The string tension of the grand unified cosmic strings is of the order $10^{-3}/G$ and thus considerably smaller than that associated with Einsteinian strings.

The simplest representatives for the minimal strings are surfaces $X^2 \times Y^2$ representable as Cartesian product of two-dimensional minimal surfaces belonging to M^4 and CP_2. Thus the equations

$$g^{\alpha\beta} H^k_{\alpha\beta} = 0 \tag{69}$$

hold separately for each factor in Cartesian product.

Since Yang-Mills energy momentum tensor is expressible in the form

$$T^{\alpha\beta} = (-g^{\alpha\beta}(X^2) - g^{\alpha\beta}(Y^2))L_{YM}/4 \tag{70}$$

its contribution to the equations of motion vanishes as a consequence of the minimal surface property. The contribution of Einstein tensor vanishes for the tensor vanishes for the same reason. Thus the equations of motion are satisfied provided the condition

$$T^{\alpha\beta} = \lambda g^{\alpha\beta}(x^2) + \lambda_1 g^{\alpha\beta}(Y^2) \tag{71}$$

is satisfied. The field equations tell nothing about the values of the "cosmological constants" λ and λ_1.

The string tension of the minimal strings is given by the expression

$$k_M = 4\pi R^2 \lambda + k \tag{72a}$$

$$k = (1/4) \int L_{\text{YM}} \sqrt{g}\, d^2x \tag{72b}$$

where L_{YM} is the symmetry broken Yang-Mills action.

Particularly interesting representatives for the minimals strings are surfaces for which the surface Y^2 is a geodesic sphere of CP_2. Since CP_2 allows two kinds of geodesic spheres (of type I and II) the minimal strings can be divided to strings of type I and type II.

One can evaluate the contribution of the Yang-Mills action to the string tension by using the explicit representation of the effective action in $U(1)$ subtheories. The general expression for the string tension is given by

$$\begin{pmatrix} k_{\text{I}} \\ k_{\text{II}} \end{pmatrix} = \begin{pmatrix} X & 1/4 \\ Y & 0 \end{pmatrix} \begin{pmatrix} 1/4g^2 \\ f \end{pmatrix} 8\pi/R^2 \tag{73a}$$

The quantities X and Y are given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \sum_{e=+1,-1} d(e) \begin{pmatrix} (31+n_e^2)/2 \\ 4 \end{pmatrix} \tag{73b}$$

The sum runs over the H chiralities and the factor $d(e)$ equals 1 or 0 depending on whether the fermions with the chirality e are elementary spinor fields.

The identification of the strings of type II as hadronic strings is possible. The vanishing of the string tension at long length scales $(1/g^2 = 0)$ implies that only the sea particles, which we identify as ♯-condensed matter, contribute to the string tension

$$k = 4\pi\lambda R^2 \tag{74}$$

The resulting string tension is of the correct order of magnitude if the value of the "cosmological constant" is of the order of

$$\lambda = 1/R^2 R_{H^2} \tag{75}$$

where R_H^2 is of the order of baryonic Compton length.

The length scale determined by the "cosmological constant" λ is roughly the geometric mean of baryonic Compton length and Planck length (within factors of 10)

$$L = (R_H R)^{1/2} \tag{76a}$$

It is amusing that the "size of the electron" obtained by setting the order of magnitude for the leptonic Higgs field equal to the electron mass

$$\text{Higgs} \cong R/L_e^2 = m_e \tag{76b}$$

is also a geometric mean of the leptonic Compton length and of CP_2 radius

$$L_e \cong (R/m_e)^{1/2} \tag{77}$$

The strings of type I have enormous string tension resulting from the Yang–Mills part of the action and resemble Einsteinian and thus cosmic strings. We shall discuss the possible cosmological role of the cosmic strings in the last section of the paper.

3.4.3. Massless Extremals

The massless extremals form a very general family of extremals characteristic to Yang–Mills action (one could add the curvature squared term to the action without losing these extremals). Let p and e be two M^4 vectors satisfying the conditions

$$\begin{aligned} p \cdot p &= 0 \\ p \cdot e &= 0 \end{aligned} \tag{78}$$

These vectors can be interpreted as four-momentum and polarization vectors. Let $F^k(x, y)$ be four arbitrary functions of the variables

$$x = p \cdot m, \qquad y = e \cdot m \tag{79a}$$

Then the surfaces defined by the conditions

$$s^k = F^k(x, y) \tag{79b}$$

are extremals of the action.

The field equations are satisfied because the energy momentum tensor, Einstein tensor, and gauge current are proportional to the quantities $k^\alpha k^\beta$ and k^α, respectively (in the standard coordinates for M^4) so that the various contractions appearing in the field equations vanish. For the same reason the solutions are massless.

In fact the assumption about the representability as a graph of some map $M^4 \to CP_2$ is not essential. A generalization of the conditions defining the solutions is

$$F^k(x, y, s) = 0, \qquad i = 1, \ldots, 4 \tag{80}$$

where F^k are four arbitrary functions of their arguments with the property that the coordinate variables s^k can be solved as finitely many-valued functions of x and y locally.

3.4.4. Minimal Surface Extremals

We have already found that stringlike minimal surfaces might provide a natural phenomenological description of hadrons at certain length scales. Since minimal surfaces can be regarded as a direct generalization of a geodesic line, which is a one-dimensional minimal surface, the minimal surface phenomenology might work also in case of other elementary particles. The minimal surface extremals might even dominate in the functional integral in the short length scale limit of the theory, when there is no sense to speak about a continuous space-time but rather a gas of 3-particle-like 3-manifolds in the space H.

The minimal surface property implies the vanishing of the Higgs field so that the concept of the unitary gauge is not well defined for these surfaces and one cannot form $SU(2)_L$ singlets in the manner described in the first paper of the series. Thus one expects that the minimal surface phase corresponds to the $SU(2)_L$ nonconfining phase of the theory.

It is clear that the addition of the Higgs term of form

$$L = a\, \mathrm{Tr}(D_\mu \not{H} D^\mu \not{H}) + V(\not{H}) \tag{81}$$

(a is some dimensionless number) makes minimal surfaces natural extremals of the effective action. It is quite evident that Higgs term cannot be present in the action at large length scales: it is impossible to obtain the results of classical electrodynamics if the Higgs term is present.

The field equations indeed allow minimal surfaces as its extremals. A characteristic feature of these extremals is that Einstein equations can contain a "cosmological term"; field equations are satisfied provided only that the condition

$$T^{\alpha\beta} = \kappa G^{\alpha\beta} + \lambda g^{\alpha\beta} \tag{82}$$

holds. Here the "cosmological constant" λ is an arbitrary parameter. Clearly, minimal surface extremals mean a departure from the GRT.

In order to obtain a description of a free elementary particle one might also require that the flow of various quantum numbers between #-condensed phase and free particle phase is vanishing: in other words the condition

$$G^{\alpha\beta} H^k_{\alpha\beta} = 0 \tag{83}$$

is satisfied.

This condition is certainly satisfied if Einstein tensor is proportional to the metric tensor

$$G^{\alpha\beta} = \kappa g^{\alpha\beta} \tag{84}$$

If the surface in question is a Cartesian product, the condition generalizes: Einstein tensor is a linear combination of the various metric tensors.

The simplest surfaces satisfying this condition are simply pieces of flat Minkowski space. It is tempting to interpret these surfaces as hadronic bags. The motivation for the identification is that string description of hadrons is not expected to work at short length scales since the string tension becomes extremely large. Thus the bag configuration is expected to be energetically more favorable. The energy of the system is simply proportional to the volume of the bag (plus boundary terms not treated here).

A second example of particlelike surfaces are membrane-type surfaces, which are surfaces of type

$$X^4 = M^1 \times S^2 \times S^1 \tag{85}$$

Here M^1 is a timelike geodesic in M^4, S^2 is the arbitrary two-dimensional minimal surface in M^3, and the orthogonal complement of M^1 and S^1 is a geodesic of CP_2. It is well known (Spivak, 1970) that minimal surfaces in M^3 have always boundary; the vanishing of the mean curvature (sum of the two main curvatures) makes the surface negatively curved so that it cannot be closed.

The contribution of the Einstein tensor to the mass of these objects has the same form as in case of stringlike objects [equation (68b)] and is of the order of Planck mass unless the two surfaces in question are topologically two spheres with two holes or torus; in this case the contribution vanishes identically.

4. GENERAL FEATURES OF COUPLING CONSTANT EVOLUTION

The basic assumption of our semiclassical approach is that the effective action depends on length scale only via the length-scale-dependent coupling constants.

In this section we consider the general features of the coupling constant evolution and show that the requirement "The theory reproduces Maxwell electrodynamics at long length scales" implies $SU(2)_L$ confinement at long length scales in the sense that the bosonic effective action reduces to a pure Maxwell action.

Furthermore, we show that the canonical invariance of Kahler action provides an explanation for color confinement. Finally, the problem what happens in short length scales is considered.

4.1. $SU(2)_L$ Confinement at the Level of the Effective Action

As found in the previous paper the various field quantities in the unitary gauge can be regarded as $SU(2)_L$ singlets formed as composites of the normalized Higgs field and spinor and gauge boson fields. Thus the concepts of $SU(2)_L$ confinement and symmetry breaking are dual.

To illustrate the nature of this duality consider the description of the mass splittings in the two pictures. The nonvanishing expectation of the Higgs field implies the existence of a preferred gauge, the unitary gauge, in which the mass matrix defined by Higgs field is diagonal and symmetry breaking. On the other hand, the field quantities in the unitary gauge can be regarded as $SU(2)_L$ singlets and thus they only apparently form $SU(2)_L$ multiplets; thus the masses of the corresponding states need not be equal.

It is important to notice that the confinement picture makes sense only if the Higgs field is nonvanishing; for minimal surfaces (say hadronic strings) found to be natural extremals of the action at short length scales this gauge is ill defined.

The experience with gauge theories suggests that the gauge coupling associated with the $SU(2)_L$ interactions should diverge at some finite length scale if the duality holds. The result in fact follows from a mere requirement that the effective action reproduces Maxwell electrodynamics at the limit of long length scales.

In the ordinary gauge theories, where all gauge fields are primary dynamical variables this requirement is easy to satisfy. The nonvanishing Higgs expectation implies that only photons propagate over macroscopic distances. Now however all field quantities are local composites of the same primary dynamical variables and this simple scenario does not work.

As far as we know, the only way to reproduce Maxwell electrodynamics at the limit of long length scales is to assume that the effective action becomes a pure $U(1)$ action at this limit so that the Weinberg angle vanishes at this limit or equivalently, the $SU(2)_L$ coupling diverges. Indeed, as already shown, the pure $U(1)$ action reproduces Maxwell electrodynamics, when the induced metric on X^4 is approximated with a flat metric.

Even more, we expect that a transition to the "Maxwellian phase," differing radically from the "Yang-Mills phase" because of its enormous vacuum degeneracy, occurs at some finite length scale L. This length scale is expected to play an important role in understanding the masses of various particles. A great challenge for the TGD approach is to evaluate the order of magnitude for this length scale and to understand, why this length scale is so enormous as compared with the length scale provided by the size of CP_2.

An obvious counterargument against the proposed scenario is that it is in contradiction with the experimental facts. The measured value of the Weinberg angle, which is believed to correspond to the value of this parameter at the limit of long length scales, is nonvanishing: $\sin^2 \theta_\omega = 0.23$ (Weinberg, 1967; Salam, 1968; Glashow, 1961).

We do not think that this counterargument is as serious as it looks at first. Weinberg angle is a parameter related to the perturbative treatment of the electroweak interactions using gauge coupling as expansion parameter.

In TGD approach the perturbative treatment makes sense at short length scales only, say at length scales of the order of Compton length of the intermediate gauge boson. Thus the measured value of the Weinberg derived by comparing the predictions of the perturbative approach to electroweak interactions must correspond to the value of this parameter at short length scales and the paradox disappears.

Also one might argue that the idea of weak confinement is absurd; weak interactions are indeed weak! The possible divergence of the electroweak coupling at some length scale should certainly manifest itself in some manner. The point is that this divergence indeed manifests itself in a very spectacular manner since it implies color confinement; colored particles do not propagate in Maxwellian phase as we will show in the next section.

An interesting problem is whether one could reproduce the result of the GWS model both in short length and long length scales using perturbative approach. This might proceed roughly as follows:

(i) In order to handle the spin in an appropriate manner replace the external current term describing the $\sharp$-condensed matter with Dirac action. The Dirac spinors are in these approaches phenomenological fields describing particles pointlike in the length scale considered; in particular, different particle families are treated using separate Dirac spinors. Observe that Dirac action is well defined irrespective of the values of the gauge couplings.

The masses of the various fermions should be input parameters in this approach and should be interpreted in terms of the Higgs expectations associated with the very small 3-manifold describing the elementary fermion. Notice that the value of the Higgs field associated with a typical surface in Maxwellian phase has the order of magnitude $\lesssim R/L$ and is extremely small as compared with the mass of a typical lepton.

(ii) One could try the calculation of the transition amplitudes around the solutions of the Einstein–Maxwell equations with vanishing spinor fields. Simplest surfaces of this type are simply the flat submanifolds of H; in particular Minkowski space itself. Of course, the perturbation theory should be performed using the coordinate variables of CP_2 as field variables.

4.2. Color Confinement

In the sequel we shall first show that the canonical invariance of the Maxwell phase implies color confinement. Furthermore, we show that string and bag models of hadrons emerge as natural descriptions of hadrons in the appropriate length scales.

4.2.1. Canonical Invariance and Color Confinement

The characteristic features of the Maxwellian phase are the vacuum degeneracy of the action [Maxwellian phase as quantum counterpart of the spin glass phase? (Sherrington and Kirkpatrick, 1975; Kirkpatrick and Sherrington, 1978)] and the approximate canonical invariance [exact to order $O((R/L)^2)$]. The action of the canonical transformation to gauge field is equivalent to that of a $U(1)$ gauge transformation.

Canonical transformations do not leave the curvature scalar part of the action even approximately invariant (the appearance of the $1/G$ factor in the action causes this). If one however allows also the energy momentum tensor of the ♯-condensed matter to change in the canonical transformation so that Einstein equations remain true, canonical transformations become dynamical symmetries transforming to each other dynamical evolutions with identical electroweak (electromagnetic) properties.

If one requires that the whole action remains approximately invariant in the canonical transformation then one must require that the gravitational coupling and by color gravitational analogy also the color coupling becomes very large at long length scales (having perhaps a pole in some hadronic mass?). In our opinion this is not necessary for color confinement to occur.

An important consequence of the canonical invariance is that color symmetries become "semilocal." Let V, $i=1,2$ be two open sets of CP_2 satisfying the conditions (1) $V_1 \subset V_2$ and (2) the intersection $\delta V_1 \cap \delta V_2$ of the boundaries of V_i is empty.

Theorem. Given sets V; with the properties specified above there exists a canonical transformation, which equals a given color isometry inside the set V_1 and reduces to identity transformation outside the set V_2.

Proof. Since color isometries are canonical transformations one can find a Hamiltonian H, which exponentiates to the given color isometry. From the assumptions (1) and (2) it follows that there exists a real-valued continuous function

$$k: CP_2 \to R$$

which has constant value

$$k(s) = 1 \tag{86}$$

in the set V_1 and vanishes outside the set V_2. The canonical transformation with the required properties is obtained by exponentiating the Hamiltonian $H_1 = kH$. ■

For a surface X^4 representable as a graph for a map $M^4 \to CP_2$ the action of this transformation is nontrivial only in the inverse image $f^{-1}(V_2)$ of the set V_2 and is equivalent to a color rotation in the set $f^{-1}(V_1)$. If the set $f^{-1}(V_2)$ consists of several components one can clearly restrict the transformation so that it acts only in one of the components nontrivially. Moreover, one can combine several transformations of this kind associated with disjoint open sets V_i.

These transformations can differ from a unit transformation in an arbitrarily small region of X^4; in this respect they resemble local gauge transformations. There are however some important differences:

(1) For an arbitrary point belonging to $f^{-1}(V)$, there exists an open set, where the transformation is a rigid color rotation. One can say that color invariance is "semilocal."

(2) These transformations are dynamically generated approximate symmetries and thus one cannot get rid of the associated "gauge degeneracy" by using the gauge-fixing procedure.

The main result of this section is that the phenomenon of color confinement can be understood as a consequence of the "semilocality" of the color invariance in the Maxwellian Phase (!). To derive this result we study the properties of the current-current correlation function (c.c.c.f.)

$$\langle J(m_1)J(m_2)\rangle \qquad (m_1 \text{ and } m_2 \text{ are points of } M^4) \tag{87}$$

associated with two color currents and defined as a functional average of the product of the classical color currents.

We do this because we expect this function to carry essential information about the properties of the color interaction; for instance in QED the scattering cross section for two charged particles is closely related to c.c.c.f.

We expect the color confinement to manifest itself via the vanishing of the c.c.c.f. for space-time intervals large compared with the length scale L, where the transition to Maxwellian phase occurs.

Consider now the evaluation of c.c.c.f. What we want to evaluate is the color correlation function in a state with given electroweak quantum numbers. The calculation neglects quantum fluctuations completely; the color correlation function is evaluated as expectation value over classical field configurations with fixed electroweak currents. In particular, weak currents associated with the #-condensed matter vanish identically by Einstein-Maxwell equations.

By color gravitational analogy the color currents are closely related to the energy momentum tensor of the #-condensed matter and one cannot pose any restriction on the energy momentum tensor of the #-condensed matter. The assumption about the color gravitational analogy is essential; otherwise one should fix also the energy momentum tensor of the #-condensed matter and the canonical degeneracy would be lost.

The triviality of the correlation function follows basically from the canonical invariance of the Maxwell action. One must average over a very large set of electromagnetically equivalent configurations and the semilocal color symmetry implies that the direction of color current in Lie algebra of $SU(3)$ is practically random for a configuration having given electroweak properties and as a consequence color correlation vanishes.

Observe that weak confinement is necessary to hinder the propagation of color. If the weak coupling were not infinite then fixing of electroweak gauge currents would fix completely the surface X^4 as well as the energy momentum tensor of the #-condensed matter by Einstein equations and the correlation function were simply the product of the classical color currents.

The triviality of color correlations in the #-condensed phase together with the uncertainty principle suggests that colored particles are very heavy in Maxwellian phase and thus do not propagate. This idea is supported also by a direct dimensional argument. The color current is given by

$$J^{\alpha}_{i)} = \kappa G^{\alpha\beta} s^{k}_{1\beta} s_{kl} j^{l}_{i)} \tag{88}$$

(color gravitational analogy!) so that one obtains the following estimate for the color charge in terms of the mass of the particle

$$q \cong MR^2/L \tag{89}$$

where L is the length scale in question. Clearly, a color charge of order unity implies a mass of the order of Planck mass.

In the preceding discussion we have made no explicit assumptions about the length scale evolution of the color and gravitational couplings; the mere requirement that $SU(2)_L$ coupling diverges at some length scale makes color propagation impossible at larger length scales. This result is not so surprising as it first looks since all field quantities are in TGD approach composites of the same primary variables.

If one accepts exact color gravitational analogy embodied in the effective action approach one expects that the linear relationship between color and gravitational couplings holds at all length scales. Thus one has two alternative scenarios for the evolution of the color coupling:

(i) No divergence appears in the couplings at hadronic length scales.

(ii) Both color and gravitational couplings have divergence at some hadronic length scale; the divergence of the gravitational coupling might be related to the existence of spin-2 hadrons.

We regard the first alternative as more natural since field equations indeed allow stringlike objects (minimal strings) having string tension as free parameter. If one requires that strings obey Einstein equations then string tension is of the order of $1/G$ and one must require that in hadronic length scales gravitational coupling becomes very large.

There is of course a scenario in which color gravitational analogy is given up. This means the addition of a separate color term in the effective action and the assumption that this term vanishes at large length scales; thus also the color coupling must diverge at some length scale.

Summarizing, the scenario in which color gravitational analogy is exact and color confinement is caused by the divergence of $SU(2)_L$ coupling seems to be the simplest description of color confinement.

4.2.2. String and Bag Pictures of Hadrons

The effective action allows stringlike minimal surfaces as its extremals as already found. The minimal surface property of these extremals makes their interpretation as hadronic strings possible.

The first consequence of the minimal surface property is that Einstein equations need not hold true and the energy momentum tensor satisfies the constraint

$$T^{\alpha\beta} = T_{\#}^{\alpha\beta} + T_{V}^{\alpha\beta} = \lambda g^{\alpha\beta}(x^2) + \lambda_1 g^{\alpha\beta}(y^2) \tag{90}$$

The parameter is not fixed by the field equations; a natural expectation is that it gives a correct hadronic string tension. The contribution of the #-condensed matter (sea particles and possible valence quarks) to string tension must be positive and the contribution of electroweak (!) monopole field (ends of the string correspond to magnetic monopoles) rapidly grows to value of the order $1/G$. Therefore these strings can have hadronic string tension in a rather limited range of length scales. In the shorter length scales stringlike configurations are not expected to dominate for the simple reason that the enormous string tension makes them energetically unfavored.

A second consequence of the minimal surface property is the vanishing of the Higgs field. Thus one expects $SU(2)_L$ nonconfinement and color nonconfinement to result. This seems to be the case also for the minimal strings in the Maxwellian phase. The point is that the argument leading to the semilocal color invariance and thus to a short correlation length for color currents does not apply to stringlike objects since any semilocal color rotation in CP_2 induces the same color rotation at all points of the string. Thus the colors associated with different points of the string are correlated.

As already noticed the string picture applies only in some rather limited length scale. It is natural to expect that minimal surfaces dominate the functional integral also in the shorter length scales. What makes the string description inapplicable is the enormous string tension deriving from electroweak monopole field. Thus the surfaces representable as graphs of a map $M^4 \rightarrow CP_2$ probably give the dominant contribution to the functional integral.

The simplest representative for this kind of a surface is a piece of Minkowski space. The energy density of the #-condensed matter is expected to be given by the expression

$$T_{\#}^{\alpha\beta} = \lambda g^{\alpha\beta} \tag{91}$$

where the parameter λ remains undetermined. One obtains an upper bound for this parameter by comparing the resulting mass with the hadronic masses.

This implies that the bag model picture (another geometric model of hadrons! (Johnson, 1975) becomes a more appropriate description of hadrons. Applying perturbative approach to the calculation of the functional integral around Minkowski-space-like vacuum extremals one obtains a formalism resembling perturbative field theory. This result is in accordance with the idea of the asymptotic freedom (Politzer, 1974) at high energies.

Combining the result obtained one can indeed understand some of the mysterious-looking features of color confinement naturally. Consider the description of a typical high-energy hadronic reaction in this scenario. Since very high energies are involved in the collision a natural description for the reaction is using effective action at short enough length scales. Thus one must give up the ordinary picture of continuous 3-space.

Rather one must describe the matter as a gas of 3-manifolds moving freely above some region (reaction volume) of classical 3-space. As the reaction proceeds reaction products #-condense continually to the underlying continuous 3-space. Because colored states do not propagate in Maxwellian phase only color singlets can escape from the reaction volume. Thus the reaction continues until all matter has left the reaction region as #-condensed color singlets.

5. COSMOLOGICAL CONSIDERATIONS

As already found the homogenous and isotropic cosmologies reduce in the limit of a vanishing mass density to M_+^4, the light cone of Minkowski space. If one accepts the breaking of Poincaré invariance in the cosmological scale then the choice $V = M_+^4$ must be considered seriously. In the following we shall show that for all globally imbeddable cosmologies the mass density is smaller than the so-called critical mass density.

The main departure of TGD approach from GRT cosmology is the presence of two forms of matter. The #-condensed matter corresponds to matter of GRT since it satisfies Einstein equations. The matter consisting of free, non-#-condensed particles has no counterpart in GRT. We shall show that this departure might provide a natural solution to the basic puzzles of the GRT cosmology provided the choice $V = M_+^4$ is accepted.

Also a model for spiral galaxies based on the properties of the cosmic strings is proposed providing a possible solution to the problem of dark matter (Fall and Lynden-Bell, 1981); the dark matter corresponds to the enormous mass density associated with cosmic strings.

5.1. The Condition $\rho < \rho_{cr}$ as Imbeddability Condition

A common feature of the M^4- and M^4_+-based cosmologies is that they both allow global imbedding only for the hyperbolic cosmologies having the property that the mass density is lower than the critical mass density (Misner et al., 1975; Adler et al., 1975)

$$\begin{aligned} \rho_{cr} &= 3H^2/8\pi G \\ H &= \dot{a}/a = 1/(g_{00})^{1/2}a \end{aligned} \tag{92}$$

To construct the imbedding to M^4_+ (and at same time to M^4) let us first restate the representation of the line element of M^4_+

$$\begin{aligned} ds^2 &= da^2 - a^2(K\,dr^2 + r^2\,d\Omega^2) \\ K &= 1/(1+r^2) \end{aligned} \tag{93}$$

Here the coordinates (a, r, θ, Φ) are related to the Minkowski coordinates (m^0, r_M, θ, Φ) by

$$\begin{aligned} a^2 &= (m^0)^2 - r_M^2 \\ ra &= r_M \end{aligned} \tag{94}$$

Clearly any surface representable as a graph for a map

$$s^k = f^k(a) \tag{95a}$$

gives rise to a hyperbolic cosmology with the metric given by

$$\begin{aligned} ds &= A\,da^2 - a^2(K\,dr^2 + r^2\,d\Omega^2) \\ A &= 1 - f^k_{1a} f^l_{1a} s_{kl} \end{aligned} \tag{95b}$$

One can imbed the cosmologies satisfying the conditions $\rho = \rho_{cr}$ and $\rho > \rho_{cr}$ for which the factor K is given by

$$K = 1 \tag{96}$$

and by

$$K = 1/(1-r^2) \tag{97}$$

respectively, only for special metrics and only partially.

The imbedding is given by the following equations:

$$\sin\theta = ka/R$$
$$\phi(r) = \int [(K - 1/(1+r^2)]^{1/2}\, dr/k \tag{98}$$

Here θ and ϕ denote the standard spherical coordinates of the geodesic sphere of type (II). The first condition can be satisfied only for times smaller than $a = R/k$ so that the imbedding cannot be global. The reason for the nonimbeddability of nonhyperbolic cosmologies is the hyperbolic nature of M^4_+.

Yang-Mills action allows as its vacuum extremal any surface representable as a submanifold of $M^4 \times D^1$, where D^1 is an arbitrary one-dimensional submanifold of CP_2. The above-described hyperbolic cosmologies are of this type. Thus the effective action poses no conditions on allowed cosmologies provided they are hyperbolic, isotropic, and homogenous, that is, that they correspond to Lorentz invariant submanifolds of M^4_+.

It should be noticed that the imbedding of a given cosmology is highly nonunique; the choice of the submanifold D^1 is arbitrary. This can be understood by first noticing that any submanifold can be represented as a surface

$$s^k = \text{const}, \qquad k \neq r \tag{99a}$$

by choosing the coordinates suitably.

By a suitable choice of the time dependence of the coordinate s^r one can imbed arbitrary hyperbolic cosmology in $M^4 \times D^1$.

It is important to notice that the different imbeddings give rise to different cosmologies in the sense that the quantity

$$D_\alpha(G^{\alpha\beta} h^k_{1\beta}) = G^{\alpha\beta} H^k_{\alpha\beta} \tag{99b}$$

describing the flow of the various isometry charges between the ♯-condensed and free phase depends on imbedding.

Which imbedding is the correct should in principle be solved from the equations governing the interaction between the ♯-condensed and "vapor" phase. In principle one needs kinetic equations for various particle densities in order to describe their evolution in the two phases.

5.2. Minkowski Space or Its Light Cone

It is not possible to differentiate in the laboratory scale between the two possible alternatives concerning the choice of the factor V in the decomposition $H = V \times CP_2$; V can be either Minkowski space or its light cone. The following arguments however favor the choice M^4_+.

(1) This choice makes the big bang cosmology a necessity and fixes the arrow of time in cosmological scale (time reflection invariance is broken also microscopically as found in the first paper of the series).

(2) The problem of horizons (Weinberg, 1972) is one central problem of the standard cosmology. In TGD one however expects that at very early times the ♯-condensed phase to become unstable so that the vapor phase consisting of free particles (compact 3-manifolds) becomes dominant (♯ evaporation).

The evaporation is expected to occur because at high temperatures the binding energy of the ♯-condensed particle is expected to be purely gravitational and to be very nearly equal to the energy of the free particle. A simple estimate for the binding energy is given by

$$E_B = E[1-(g_{00})^{1/2}] = E(1-1/\dot{a}) \tag{100}$$

where E is the free particle energy, is expected to give an estimate for the energy of ♯-condensed particle. Thus the particles having free energy smaller than the temperature are expected to ♯ evaporate.

In fact, the stronger assumption that ♯ evaporation occurs totally at some temperature seems rather natural.

Thus one expects that at very high temperatures the cosmology reduces to light cone cosmology (M^4_+ itself can be regarded as an empty hyperbolic cosmology). Because of its flatness M^4_+ has no horizons. The absence of horizons in turn nicely solves the paradoxes associated with the isotropy of the microwave background (Weinberg, 1972); since there are no horizons the thermal equilibrium is possible in whole $a = \text{const}$ hyperboloid of M^4_+ and hence also the isotropy of M^4_+ background.

It should be emphasized that the assumption about the total transition to M^4_+ cosmology is unnecessarily strong. All that is needed to solve the horizon problem is that the vapor phase acts as a heat bath keeping the temperature of the ♯-condensed phase constant in whole a-const hyperboloid of M^4_+.

(3) A second puzzle of GRT based cosmology is the large photon/baryon ratio r (Dolgov and Zeldovich, 1981)

$$r \cong 10^9 \tag{101}$$

TGD-based cosmology might provide a natural solution to this puzzle also. The argument goes as follows.

The particles of given mass M are expected to be ♯-condenses totally, when the temperature $T \cong M$ is reached if the density of the ♯ condensate is so large that gravitational binding dominates and gravitational binding energy is approximately equal to the particle energy.

In order to understand what might happen in ♯ condensation consider the reverse of the ♯-condensation process. In ♯ evaporation a large amount of thermal energy must be supplied in order to liberate the ♯-condensed matter. This leads to a drop of pressure implying a gravitational collapse of the vapor phase so that the number density (in M^4_+) of particles considered increases. Of course, in ♯ condensation just the opposite occurs.

Since photons ♯ condense much later (on the average) and thus in much weaker gravitational field than baryons, the reduction of the photon number density is expected to be much smaller than that associated with baryons. As a result the ratio of baryon and photon number densities is reduced from its value before the ♯ condensation of baryons. Since baryons ♯-condense at a very high temperature this reduction might explain the order of magnitude for the parameter r.

A rough model of ♯ condensation in accordance with this physical picture is obtained by the replacement.

$$g_{00}(M^4_+) \to g_{00}(x^4) \tag{102}$$

in the formulas for energy density, number density, etc., this rule implies that the numerical value of energy density remains invariant in ♯-condensation, the number density of ♯ is however reduced by a factor $1/\dot{a}$

$$n \to n/\dot{a} \tag{103}$$

Let us now derive a rough estimate for the photon to baryon ratio r using this result. In order to obtain the estimate we use the expression for the temperature

$$T = T_0 a_0 / a \tag{104}$$

T_0 and a_0 can be chosen to be the present values of these quantities ($T_0 = 3\ K$; $a_0 = 10^{10}\,\text{yr}$).

Also we use Einstein equations

$$\dot{a}^2 = GKT^4 a^2 \tag{105}$$

where K is some factor of order unity for present purposes. Therefore we obtain an estimate for the expansion velocity a

$$\dot{a} = (KG)^{1/2} a_0 T_0 M_P \cong 10^6 \tag{106}$$

(M_P is the mass of proton). The estimate for the ratio r is obtained from the expression for the baryon energy density

$$\rho_B \cong pT^4 \tag{107}$$

where p is the parameter describing the matter–antimatter asymmetry

(Dolgov and Zeldovich, 1981)

$$p = (n_B - \bar{n}_B)/n_B \tag{108}$$

where the baryon and antibaryon number densities are estimated in temperature, where baryons and antibaryons are in thermal equilibrium.

The estimate for the ratio is given by

$$r \cong \dot{a}/p = 10^6/p \tag{109}$$

Clearly, the order of magnitude of this ratio is determined by the #-condensation process.

5.3. Cosmic Strings

The effective action allows as its extremals two kinds of cosmic strings: the Einsteinian and minimal strings of type I. The string tension associated with Einsteinian strings is given by

$$k = 1/4G \tag{110a}$$

The corresponding quantity for the minimal strings in the long length scale limit of the theory is given by

$$k = 8\pi f/4R^2 = (9\alpha_S/32\alpha)/4G \tag{110b}$$

Here we have used the relationships

$$f = g/16\pi\alpha \tag{111}$$

(The induced Kahler form is identified as Maxwell field in long length scales) and the relationship

$$\alpha_S = 4G/R^2 \tag{112}$$

is assumed to hold true in order to obtain the expression in terms of the gravitational coupling.

When studying the imbedding of Reissner–Nördstrom solution we derived an upper bound for the color coupling (equation (65)); the assumption that the value of the color coupling is maximal implies that the string tension of the strings of type I is 8 times larger than the string tension of the Einsteinian strings. The value of the string tension is considerably larger than that associated with the cosmic strings of the grand unified theories (Zeldovich, 1980); in fact the mass associated with a static Einsteinian string of length L is equal to the mass of a black hole of radius $L/2$.

It has been suggested that cosmic strings of the ordinary gauge theories might have acted as seeds of the galaxy formation at the early stages of the Universe (Zeldovich, 1980). This is possible only if the cosmic strings are stable enough. One however expects the enormous string tension of cosmic strings to cause a rapid collapse of the ordinary, open cosmic strings encountered in GUT's (Zeldovich, 1980) implying a far too short life time for these objects.

For the cosmic strings of TGD the enormous gravitational field near the ends of the cosmic string might hinder the decay of the cosmic string by the emission of particle radiation (note the analogy with black holes).

A particularly interesting family of cosmic strings are the objects, which might be called spiral strings. Using comoving coordinates (a, r, θ, ϕ) for the light cone of M_+^4 and coordinates (a, r) for X^2 the representation of the surface X^2 is given

$$\begin{aligned} \theta &= \pi/2 \\ \phi &= \phi(r) \end{aligned} \tag{113}$$

The equations expressing the minimal surface property of X^2 reduce to a single equation

$$\phi_{1r1r} + 2\phi_{1r}/r + (\phi_{1r})^3 r(1+r^2) = 0 \tag{114}$$

The linearization of this equation yields the equation

$$\phi_{1r1r} + 2\phi_{1r}/r = 0 \tag{115}$$

A general solution of this equation is given by

$$\phi = a + v/r \tag{116}$$

where a and v are constants. If the condition

$$v < 1$$

is satisfied, the linearization is expected to be a good approximation.

Equation (114) can be solved also exactly; taking the quantity

$$u = r^2\phi_{1r} \tag{117}$$

as a new variable equation can be cast into the form

$$u_{1r} + u^3(1+r^2)/r^3 = 0 \tag{118}$$

and thus one can represent the solution as

$$\phi(r) = \int [-1 + 2r^2 \ln(r/r_0)]^{-1/2}\, dr \tag{119}$$

The following properties of these objects make them rather interesting cosmologically:

(1) In the comoving coordinates these strings look static spiral strings in (x, y) plane. In Minkowski coordinates these strings rotate with an angular velocity

$$\omega \cong v/r_M \tag{120}$$

When the condition $v<1$ is satisfied the angular velocity is apart from logarithmic factors inversely proportional to the distance from the origin and as a consequence the orbital velocity of the string is constant in logarithmic accuracy.

(2) The equations defining the string make sense only when the condition

$$r_M \lesssim m^0 \tag{121}$$

is true and thus an upper bound for the size of these objects is given by this condition; observe that the angular velocity vanishes when this condition holds. The equations defining the string become ill defined also when the condition

$$r^2 \ln(r/r_0) < 1/2 \tag{122}$$

holds. The critical distance r_C is however extremely small for reasonable values of the parameter v (the orbital velocity of the string) given by the identification

$$v(r) = [2 \ln(r/r_0)] \tag{123}$$

(3) The energy density of the string behaves as

$$\rho \propto (r - r_C)^{-1/2} \tag{124}$$

near the singular value r_C of the radial variable. Also the amount of string per unit volume becomes very large near the origin and the average energy density becomes very large in this region.

These properties of cosmic strings motivate the following model for the structure of the spiral galaxies.

(1) Spiral (perhaps also elliptic) galaxies are concentrations of the ordinary matter around cosmic strings (spiral arms correspond to cosmic strings).

(2) The dynamics of the galaxy is determined by the motion of the cosmic string:

(a) The motion of the cosmic string is not appreciably disturbed by the presence of the ordinary matter.

(b) Ordinary matter is gravitationally bound to the cosmic string and rotates with the same angular velocity as the cosmic string in the galactic plane.

This simple model explains some of the basic properties of the spiral galaxies amazingly well as we wish to show now. First, one obtains an upper bound for the size of the stringlike structures (which can be also larger than galaxies). From the requirement (121) as a function of cosmic time. If one accepts that the oldest galaxies are formed at the time of recombination ($t \sim 10^5$ yr) this criterion gives an upper bound

$$L \lesssim 10^5 \text{ light years} \tag{125}$$

for the size of the galaxy.

The same criterion when applied for $a \simeq 10^{10}$ yr, gives the upper bound

$$L \lesssim 10^{10} \text{ light years} \tag{126}$$

for the size of the largest stringlike structures. The observations suggest the presence of stringlike structures of this size (Zeldovich et al., 1982).

Second, the model explains at least qualitatively the visible form of the galaxies. A direct test for the model is the prediction that the form of the spiral arms should be given by the equation

$$\begin{aligned} \phi &= K/r_M \\ K &= vm^0 \end{aligned} \tag{127}$$

where the parameter K is directly proportional to the age of the galaxy.

Thirdly, the average energy density associated with the string is largest in the region near the origin and this region is expected to bind ordinary matter most efficiently and thus to form the nucleus of the galaxy.

Finally, recent observations (Einasto et al., 1974; Gallagher, 1979) suggest strongly that the objects in the galactic plane move with an orbital velocity, which becomes constant at large distances from the galactic nucleus. The velocity depends on the galaxy but has the order of magnitude

$$v = 10^{-3} \tag{128}$$

The qualitative behavior is just what one expects in our model! The value of the parameter appearing in the linearized equations (116) can be identified as the orbital velocity of the matter and thus should have the order of magnitude given by equation (128). For the exact solution the velocity departs from the constant value only by slowly varying logarithmic terms.

The proposed model of spiral galaxy differs decisively from the models, which assume the existence of a galactic halo consisting of dark matter (Fall and Lynden-Bell, 1981): cosmic strings take the role of dark matter now. In these models it is assumed that the objects in the galactic plane move on approximately circular orbits so that the radius of a given orbit is given by the condition

$$v^2/R = GM(R)/R^2 \tag{129}$$

Here the quantity $M(R)$ denotes the mass inside a sphere having galactic nucleus as its center. This model explains the observations $[v(r) \cong \text{const}]$ provided the condition

$$\begin{aligned} M(R) &= aR \\ a &= v^2/G = 10^{-6}/G \end{aligned} \tag{130}$$

is satisfied.

It is amusing that our model predicts the same behavior for $M(R)$ at large distances R; the coefficient a gives an upper bound for the string tension of the cosmic string. This upper bound can be satisfied only in case of strings of type I and provided the value of color coupling at long length scales is small enough.

For the radius of CP_2 one would obtain

$$R \cong 10^3/G \tag{131}$$

so that the radius of CP_2 is of the same order as the Compton length associated with the super massive gauge bosons of GUT's (Fritsch and Minkowski, 1975; Georgi, 1975) and too large. We see however no compelling reason to require that the argument based on the use of Kepler's law should be taken seriously if the visible matter plays no essential role in the dynamics of the galaxy.

What makes the cosmic strings so exciting is that they might provide a mechanism generating the observed matter-antimatter asymmetry (Dolgov and Zeldovich, 1981).

(1) The theory breaks CP symmetry (Cronin, 1981) as found in the previous paper of this series.

(2) CP violation makes possible that the probability for a cosmic string to emit fermion differs from the probability to emit antifermion. As a consequence cosmic strings develop nonvanishing baryon and lepton numbers.

(3) Assume that the total fermion number densities vanish at all times.

These assumptions imply that cosmic strings induce matter-antimatter asymmetry in the surrounding visible matter at very early times. Therefore, when temperature becomes sufficiently small, the fermion-antifermion annihilation begins (Fitch, 1981) and leads to a Universe consisting preferentially of matter.

The proposed mechanism differs from the corresponding mechanisms proposed in the context of GUT's in that the role of massive gauge bosons is taken by cosmic strings. This mechanism could of course work in TGD also, provided leptons are assumed to be 3-quark composites. What is needed are the counterparts of the super heavy gauge bosons B having quark number 2 and having the decay modes $B \to L+q$ and $B \to \bar{q}+\bar{q}$. If the rate for the lepton (quark) production is CP-noninvariant matter-antimatter asymmetry results.

6. CONCLUSIONS AND OUTLOOK

We have devoted this paper to the following problems related to the construction of dynamics in TGD framework.

(1) To define the concepts of spacetime, particle, field, etc. as length-scale-dependent concepts.

(2) To formulate a semiclassical description of matter based on the idea that matter appears in three forms in a given length scale. The "classical matter" corresponds to #-condensed particles pointlike in the length scale considered and to the energy densities associated with the fields defined in the classical spacetime. The matter, which has not suffered #-condensation has no counterpart in GRT.

(3) To derive nontrivial information about the general properties of the theory by using the concept of the bosonic effective action and by making some general assumptions about its form.

We have found that the classical matter satisfies Einstein equations; the nonconservation of energy and related quantities can be understood as a signal about the exchange of the conserved quantities between the classical matter and free particles.

The assumption of the minimally broken conformal invariance of the length-scale-dependent matter effective action together with some technical assumptions led to the symmetry broken Yang-Mills action with length-scale-dependent coupling constants as a unique candidate for the matter part of the effective action.

The requirement that effective action reproduces Maxwell electrodynamics at long length scales leads to the $SU(2)_L$ confinement picture, which was in the previous paper found to be dual with the symmetry-breaking picture. Furthermore, the transition to the Maxwellian phase, where the action reduces to a pure Maxwell action and where the $SU(2)_L$ coupling diverges, probably occurs at some finite length scale L, which should play an important role in determining the elementary particle masses.

The feature uniquely distinguishing the Einstein-Maxwell action of TGD from the ordinary Einstein-Maxwell action is its enormous vacuum degeneracy resulting from the approximate canonical invariance of Maxwell action. It was found that the canonical invariance implies "semilocalization" of the color symmetries. As a consequence the current-current correlation functions for color currents are expected to be very short ranged. We regard this as a symptom of color confinement. Of course, the very occurrence of $SU(2)_L$ confinement and its close relationship with color confinement is a feature characteristic to TGD approach.

We found that at short length scales the concept of the classical space-time probably does not make sense; rather one must describe the matter as a gas of particlelike 3-manifolds moving in Minkowski space or its light cone. This result was found to have important cosmological consequences.

Summarizing, to our opinion the results of this and earlier papers show that TGD approach to the description of the fundamental interactions should be taken seriously. It must of course be admitted that there are many weakly understood issues. To mention only one. How to understand the enormous size of the elementary particle length scale as compared with the length scale of the space CP_2?

An important task to be faced is the construction of quantum TGD. Although the argumentation based on the concept of length-scale-dependent effective action has produced nice results, the task of actually performing functional integral over submanifolds of H seems horrible. Moreover, we have no formal proof for the unitarity of the possibly resulting theory. Thus a deep revision of the existing ideas about the construction of quantum theory may well be needed in order to build a proper quantum TGD. We shall return to this problem in the third paper of series.

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APPENDIX. BASIC PROPERTIES OF CP_2

A1. CP_2 as a Manifold

CP_2, the complex projective 2-space, is defined by identifying the points of the complex 3-space $\mathbb{C}^3$ under the equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) \tag{A1}$$

Here λ is any nonzero complex number. The pair z^i/z^j for a fixed j and $zi \neq 0$ defines a complex coordinate chart for CP_2. As j runs from 1 to 3 one obtains an atlas of three charts covering CP_2 the charts being holomorphically related to each other (e.g., CP_2 is a complex manifold). The points $z^3 \neq 0$ form a subset of CP_2 homeomorphic to R^4 and the points with $z^3 = 0$ a set homeomorphic to $CP_1 = S^2$. Therefore CP_2 is obtained from R^4 by "adding the 2-sphere at infinity."

Besides the complex coordinates $\xi^2 = z^i/z^3$, $i = 1, 2$, the coordinates of Eguchi and Freund (Eguchi et al., 1980) will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it \\ \xi^2 &= x + iy \end{aligned} \tag{A2}$$

These are related to "spherical" coordinates via the equations

$$\begin{aligned} \xi^1 &= r \exp[i(\psi + \phi)/2] \cos(\theta/2) \\ \xi^2 &= r \exp[i(\psi - \phi)/2] \sin(\theta/2) \end{aligned} \tag{A3}$$

The ranges of the variables r, θ, ϕ, and ψ are $[0, \infty]$, $[0, \pi]$, $[0, 4\pi]$, and $[0, 2\pi]$, respectively.

Considered as a real four-dimensional manifold CP_2 is compact and simply connected, with Euler number 3, Pontryagin number 3, and second Betti number $b = 1$. The last property stems from the fact that the second homology group $H_2(CP_2)$ is isomorphic to integers.

A2. Metric and Kahler Structures of CP_2

In order to obtain a natural metric for CP_2 observe that CP_2 can be thought of as a set of the orbits of the isometries $z^i \to \exp(i\alpha) z^i$ on the sphere S^5: $\Sigma(z_i|^2 = R^2$. The metric of CP_2 is obtained by projecting the metric of S^5 orthogonally to the orbits. Therefore the distance between the points of CP_2 is that between the representative orbits on S^5. The line element has the following form in the complex coordinates:

$$ds^2 = g_{a\bar{b}}\, d\xi^a\, d\bar{\xi}^b \tag{A4}$$

where the Hermitian metric $g_{a\bar{b}}$ is defined by

$$g_{a\bar{b}} = R^2\, \partial a\, \partial \bar{b} \ln F \tag{A5}$$

The quantity F is defined as

$$F = 1 + r^2 \tag{A6}$$

An explicit representation of the metric is given by

$$ds^2/R^2 = (dr^2 + r^2\sigma_3^2)/F + r^2(\sigma_1^2 + \sigma_2^2)/F^2 \tag{A7}$$

where the quantities σ_i are defined as

$$\begin{aligned} &r^2\sigma_1 = \mathrm{Im}(\xi^1\, d\xi^2 - \xi^2\, d\xi^1), \qquad r^2\sigma_2 = -\mathrm{Re}(\xi^1\, d\xi^2 - \xi^2\, d\xi^1) \\ &r^2\sigma_3 = -\mathrm{Im}(\Sigma\xi^k\, d\bar{\xi}^k) \end{aligned} \tag{A8}$$

The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 e_k^A e_l^B\, \delta_{A,B} \tag{A9}$$

are given by

$$\begin{aligned} &e^0 = dr/F, \qquad e^1 = r\sigma_1/\sqrt{F} \\ &e^2 = r\sigma_2/\sqrt{F}, \qquad e^3 = r\sigma_3/F \end{aligned} \tag{A10}$$

The vierbein connection satisfying the defining relation

$$de^A = -V_B^A \wedge e^B \tag{A11}$$

is given by

$$\begin{aligned} &V_{01} = -e^1/r, \qquad V_{23} = e^1/r \\ &V_{02} = -e^2/r, \qquad V_{31} = e^2/r \\ &V_{03} = (r - 1/r)e^3, \qquad V_{12} = (2r + 1/r)e^3 \end{aligned} \tag{A12}$$

The representation of the curvature (components of the curvature tensor in the vielbein basis) are constant reflecting the fact CP_2 is a constant curvature space:

$$\begin{aligned} &R_{01} = e^0 \wedge e^1 - e^2 \wedge e^3, \qquad R_{23} = -e^0 \wedge e^1 + e^2 \wedge e^3 \\ &R_{02} = e^0 \wedge e^2 - e^3 \wedge e^1, \qquad R_{31} = -e^0 \wedge e^2 + e^3 \wedge e^1 \\ &R_{03} = 4e^0 \wedge e^3 + 2e^1 \wedge e^2, \qquad R_{12} = 2e^0 \wedge e^3 + 4e^1 \wedge e^2 \end{aligned} \tag{A13}$$

The metric defines a real, covariantly constant, and therefore closed 2-form J:

$$J = -ig_{a\bar{b}}\, d\xi^a\, d\bar{\xi}^b \tag{A14}$$

Because J is closed, CP_2 by definition is a Kahler manifold. The Kahler form J defines in CP_2 a symplectic structure because it satisfies

$$J_l^k J_r^l = -\delta_r^k \tag{A15}$$

The form $2J$ is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a $U(1)$

gauge potential B carrying a magnetic charge of unit $1/29$ (g denotes the gauge coupling). Locally one has therefore

$$J = dB \tag{A16}$$

where B is the so-called Kahler potential.

It should be noticed that the magnetic flux of J through a 2-surface in CP_2 is proportional to its homology equivalence class, which is integer valued. The explicit representation of J and B are given by

$$\begin{aligned} B &= 2re_3 \\ J &= 2(e_0 \wedge e_3 + e_1 \wedge e_2) \end{aligned} \tag{A17}$$

The vielbein curvature form and Kahler form that are in complex coordinates are covariantly constant and in complex coordinates they have only components of type $V_{a\bar{b}} = -V_{\bar{b}a}$ and $J_{a\bar{b}} = -J_{\bar{b}a}$, respectively ($V_{ab} = V_{\bar{a}\bar{b}} = 0$ and $J_{ab} = J_{\bar{a}\bar{b}} = 0$).

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